Presentations for monoids of finite partial isometries

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Abstract

In this paper we give presentations for the monoid $\mathcal{DP}_n$ of all partial isometries on $\{1,\ldots,n\}$ and for its submonoid $\mathcal{ODP}_n$ of all order-preserving partial isometries.

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Introduction

Semigroups of order-preserving transformations have long been considered in the literature. A short, and by no means comprehensive, history follows. In 1962, Aĭzenštat [1] and Popova [25] exhibited presentations for $\mathcal{O}_n$, the monoid of all order-preserving full transformations on a chain with n elements, and for $\mathcal{PO}_n$, the monoid of all order-preserving partial transformations on a chain with n elements. Some years later, in 1971, Howie [21] studied some combinatorial and algebraic properties of $\mathcal{O}_n$ and, in 1992, Gomes and Howie [18] revisited the monoids $\mathcal{O}_n$ and $\mathcal{PO}_n$. Certain classes of divisors of the monoid $\mathcal{O}_n$ were determined by Higgins [19] in 1995 and by Fernandes [8] in 1997. More recently, Laradji and Umar [23, 24] presented more combinatorial properties of these two monoids. The injective counterpart of $\mathcal{O}_n$, i.e. the monoid $\mathcal{POI}_n$ of all injective members of $\mathcal{PO}_n$, has been object of study by the first author in several papers [8, 9, 10, 12, 13], by Derech in [5], by Cowan and Reilly in [4], by Ganyushkin and Mazorchuk in [15], among other authors. Presentations for the monoid $\mathcal{POI}_n$ and for its extension $\mathcal{PODI}_n$, the monoid of all injective order-preserving or order-reversing partial transformations on a chain with n elements, were given by Fernandes [10] in 2001 and by Fernandes et al. [14] in 2004, respectively. See also [11], for a survey on known presentations of transformations monoids. We notice that the first author together with Delgado [6, 7] have computed the abelian kernels of the monoids $\mathcal{POI}_n$ and $\mathcal{PODI}_n$, by using a method that is strongly dependent of given presentations of the monoids.

The study of semigroups of finite partial isometries was initiated by Al-Kharousi et al. in [2, 3]. The first of these two papers is dedicated to investigate some combinatorial properties of the monoid $\mathcal{DP}_n$ of all partial isometries on $\{1,\ldots,n\}$ and of its submonoid $\mathcal{ODP}_n$ of all order-preserving (considering the usual order of $\mathbb{N}$) partial isometries, in particular, their cardinalities. The second one presents the study of some of their algebraic properties, namely Green's structure and ranks. Recall that the *rank* of a monoid M is the minimum of the set $\{|X| \mid X \subseteq M \text{ and } X \text{ generates } M\}$. See e.g. [20] for basic notions on Semigroup Theory.

The main aim of this paper is to exhibit presentations for the monoids $\mathcal{DP}_n$ and $\mathcal{ODP}_n$. We would like to point out that we made considerable use of computational tools, namely, of GAP [16].

Next, we introduce precise definitions of the objects considered in this work.

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Let $n \in \mathbb{N}$ and $X_n = \{1, \ldots, n\} \subset \mathbb{N}$ (with the usual arithmetic and order). Let $\mathcal{I}_n$ be the symmetric inverse semigroup on X_n, i.e. the monoid, under composition of maps, of all partial permutations of X_n.

Let $\alpha \in \mathcal{I}_n$. We say that α is *order-preserving* (respectively, *order-reversing*) if, for all $i, j \in \mathrm{Dom}(\alpha)$, $i \leq j$ implies $i\alpha \leq j\alpha$ (respectively, $i\alpha \geq j\alpha$). Clearly, the product of two order-preserving transformations or two order-reversing transformations is an order-preserving transformation and the product of an order-preserving transformation by an order-reversing transformation, or vice-versa, is an order-reversing transformation. On the other hand, we say that α is an *isometry* if, for all $i, j \in \mathrm{Dom}(\alpha)$, $|i\alpha - j\alpha| = |i - j|$.

Define

$$\mathcal{PODI}_n = \{\alpha \in \mathcal{I}_n \mid \alpha \text{ is order-preserving or order-reversing}\},$$

$$\mathcal{POI}_n = \{\alpha \in \mathcal{I}_n \mid \alpha \text{ is order-preserving}\},$$

$$\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n \mid \alpha \text{ is an isometry}\}$$

and

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{I}_n \mid \alpha \text{ is an order-preserving isometry}\}.$$

All these sets are inverse submonoids of $\mathcal{I}_n$ (see [14, 3]). Obviously, $\mathcal{POI}_n \subseteq \mathcal{PODI}_n$ and $\mathcal{ODP}_n = \mathcal{DP}_n \cap \mathcal{POI}_n$. Moreover, as observed by Al-Kharousi et al. [3], we also have $\mathcal{DP}_n \subseteq \mathcal{PODI}_n$.

For simplicity, from now on we consider $n \geq 3$.

1 Preliminaries

Let X be a set and denote by X^* the free monoid generated by X. A *monoid presentation* is an ordered pair $\langle X \mid R \rangle$, where X is an alphabet and R is a subset of $X^* \times X^*$. An element (u, v) of $X^* \times X^*$ is called a *relation* and it is usually represented by $u = v$. To avoid confusion, given $u, v \in X^*$, we will write $u \equiv v$, instead of $u = v$, whenever we want to state precisely that u and v are identical words of X^*. A monoid M is said to be *defined by a presentation* $\langle X \mid R \rangle$ if M is isomorphic to X^*/ρ_R, where ρ_R denotes the smallest congruence on X^* containing R. For more details see [22] or [26].

Given a finite monoid T, it is clear that we can always exhibit a presentation for it, at worse by enumerating all its elements, but clearly this is of no interest, in general. So, by finding a presentation for a finite monoid, we mean to find in some sense a *nice* presentation (e.g. with a small number of generators and relations).

A usual method to find a presentation for a finite monoid is described by the following result, adapted for the monoid case from [26, Proposition 3.2.2].

Theorem 1.1 (Guess and Prove method) *Let M be a finite monoid, let X be a generating set for M, let $R \subseteq X^* \times X^*$ be a set of relations, and let $W \subseteq X^*$. Assume that the following conditions are satisfied:*

1. *The generating set X of M satisfies all the relations from R;*
2. *For each word $w \in X^*$, there exists a word $w' \in W$ such that the relation $w = w'$ is a consequence of R;*
3. *$|W| \leq |M|$.*

Then, M is defined by the presentation $\langle X \mid R \rangle$.

Notice that, if W satisfies the above conditions then, in fact, $|W| = |M|$.

Let X be an alphabet, $R \subseteq X^* \times X^*$ a set of relations and W a subset of X^*. We say that W is a set of *forms* for the presentation $\langle X \mid R \rangle$ if the condition 2 of Theorem 1.1 is satisfied. Suppose that the empty word belongs to W and, for each letter $x \in X$ and for each word $w \in W$, there exists a word $w' \in W$ such that the relation $wx = w'$ is a consequence of R. Then, it is easy to show that W is a set of forms for $\langle X \mid R \rangle$.

Given a presentation for a monoid, another method to find a new presentation consists in applying Tietze transformations. For a monoid presentation $\langle A \mid R \rangle$, the four *elementary Tietze transformations* are:

(T1) Adding a new relation $u = v$ to $\langle A \mid R\rangle$, providing that $u = v$ is a consequence of R;

(T2) Deleting a relation $u = v$ from $\langle A \mid R\rangle$, providing that $u = v$ is a consequence of $R\backslash\{u = v\}$;

(T3) Adding a new generating symbol b and a new relation $b = w$, where $w \in A^*$;

(T4) If $\langle A \mid R\rangle$ possesses a relation of the form $b = w$, where $b \in A$, and $w \in (A\backslash\{b\})^*$, then deleting b from the list of generating symbols, deleting the relation $b = w$, and replacing all remaining appearances of b by w.

The next result is well-known (e.g. see [26]):

Theorem 1.2 *Two finite presentations define the same monoid if and only if one can be obtained from the other by a finite number of elementary Tietze transformations* $(T1)$, $(T2)$, $(T3)$ *and* $(T4)$.

As for the rest of this section, we will describe a process to obtain a presentation for a finite monoid T given a presentation for a certain submonoid of T. This method was developed by Fernandes et al. in [14] and applied there to construct a presentation, for instance, for the monoid $\mathcal{PODI}_n$. Here we will apply it to deduce a presentation for $\mathcal{DP}_n$.

Let T be a (finite) monoid, S be a submonoid of T and y an element of T such that $y^2 = 1$. Let us suppose that T is generated by S and y. Let $X = \{x_1, \ldots, x_k\}$ ($k \in \mathbb{N}$) be a generating set of S and $\langle X \mid R\rangle$ a presentation for S. Consider a set of forms W for $\langle X \mid R\rangle$ and suppose there exist two subsets W_α and W_β of W and a word $u_0 \in X^*$ such that $W = W_\alpha \cup W_\beta$ and u_0 is a factor of each word in W_α. Let $Y = X \cup \{y\}$ (notice that Y generates T) and suppose that there exist words $v_0, v_1, \ldots, v_k \in X^*$ such that the following relations over the alphabet Y are satisfied by the generating set Y of T:

$(N\!R_1)$ $yx_i = v_iy$, for all $i \in \{1, \ldots, k\}$;

$(N\!R_2)$ $u_0y = v_0$.

Observe that the relation (over the alphabet Y)

$(N\!R_0)$ $y^2 = 1$

is also satisfied (by the generating set Y of T), by hypothesis.

Let

$$\overline{R} = R \cup N\!R_0 \cup N\!R_1 \cup N\!R_2 \quad \text{and} \quad \overline{W} = W \cup \{wy \mid w \in W_\beta\} \subseteq Y^*.$$

Under these conditions, Fernandes et al. [14, Theorem 2.4], proved:

Theorem 1.3 *If W contains the empty word then $\overline{W}$ is a set of forms for the presentation $\langle Y \mid \overline{R}\rangle$. Moreover, if $|\overline{W}| \leq |T|$ then the monoid T is defined by the presentation $\langle Y \mid \overline{R}\rangle$.*

2 Some properties of the monoids $\mathcal{ODP}_n$ and $\mathcal{DP}_n$

The cardinals (among other combinatorial properties) of the monoids $\mathcal{ODP}_n$ and $\mathcal{DP}_n$ were computed by Al-Kharousi et al. in [2]. They showed that

$$|\mathcal{ODP}_n| = 3 \cdot 2^n - 2(n+1) \quad \text{and} \quad |\mathcal{DP}_n| = 3 \cdot 2^{n+1} - (n+2)^2 - 1.$$

Next, we present another (short) proof of these equalities that, in our opinion, gives us more insight.

Let $\mathcal{I}_n^+ = \{\alpha \in \mathcal{I}_n \mid i \leq i\alpha\}$ (extensive partial permutations), $\mathcal{I}_n^- = \{\alpha \in \mathcal{I}_n \mid i\alpha \leq i\}$ (co-extensive partial permutations), $\mathcal{ODP}_n^+ = \mathcal{ODP}_n \cap \mathcal{I}_n^+$ and $\mathcal{ODP}_n^- = \mathcal{ODP}_n \cap \mathcal{I}_n^-$. On the other hand, it is easy to check that

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{I}_n \mid i\alpha - i = j\alpha - j, \text{ for all } i, j \in \mathrm{Dom}(\alpha)\}.$$

Then, clearly,

$$\mathcal{ODP}_n = \mathcal{ODP}_n^+ \cup \mathcal{ODP}_n^- \quad \text{and} \quad E(\mathcal{I}_n) = \mathcal{ODP}_n^+ \cap \mathcal{ODP}_n^- \ ,$$

where $E(\mathcal{I}_n)$ denotes the set of all idempotents of $\mathcal{I}_n$, which is formed by all partial identities of X_n. Furthermore, given a nonempty subset X of X_n, in $\mathcal{ODP}_n$, we have exactly $\min(X)$ co-extensive transformations with domain X and $n - \max(X) + 1$ extensive transformations with domain X. For example, the elements of $\mathcal{ODP}_9$ with domain $\{3, 5, 6\}$ are

$$\begin{pmatrix} 3 & 5 & 6 \\ 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 6 \\ 2 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 6 \\ 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 3 & 5 & 6 \\ 5 & 7 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & 5 & 6 \\ 6 & 8 & 9 \end{pmatrix}$$

(respectively, 3 co-extensive and 4 extensive transformations). Since the number of (nonempty) subsets of X_n with minimum equal to k is 2^{n-k} and the number of (nonempty) subsets of X_n with maximum equal to k is 2^{k-1}, for $1 \leq k \leq n$, we may deduce that

$$|\mathcal{ODP}_n^-| = 1 + \sum_{k=1}^{n} k 2^{n-k} \quad \text{and} \quad |\mathcal{ODP}_n^+| = 1 + \sum_{k=1}^{n} (n - k + 1) 2^{k-1}.$$

On the other hand, it a routine to show that $1 + \sum_{k=1}^{n} k 2^{n-k} = 2^{n+1} - (n + 1) = 1 + \sum_{k=1}^{n} (n - k + 1) 2^{k-1}$, whence

$$|\mathcal{ODP}_n^-| = |\mathcal{ODP}_n^+| = 2^{n+1} - (n + 1).$$

Finally, since $|E(\mathcal{I}_n)| = 2^n$, we get

$$|\mathcal{ODP}_n| = |\mathcal{ODP}_n^-| + |\mathcal{ODP}_n^+| - |E(\mathcal{I}_n)| = 2(2^{n+1} - (n + 1)) - 2^n = 3 \cdot 2^n - 2(n + 1).$$

Now, let

$$h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} \in \mathcal{DP}_n.$$

Notice that the identity (of X_n) and h are the only permutations of $\mathcal{DP}_n$. On the other hand, given $\alpha \in \mathcal{DP}_n$, it is clear that α is an order-reversing transformation if and only if $h\alpha$ (and αh) is an order-preserving transformation (see [14]). Hence, as $\alpha = h^2 \alpha = h(h\alpha)$, it follows that the monoid $\mathcal{DP}_n$ is generated by $\mathcal{ODP}_n \cup \{h\}$. Moreover, it is also easy to deduce that

$$\mathcal{DP}_n = \mathcal{ODP}_n \cup h \cdot \mathcal{ODP}_n \quad \text{and} \quad \mathcal{ODP}_n \cap h \cdot \mathcal{ODP}_n = \{\alpha \in \mathcal{I}_n \mid |\operatorname{Dom}(\alpha)| \leq 1\}.$$

Thus

$$\begin{array}{rcl} |\mathcal{DP}_n| & = & |\mathcal{ODP}_n| + |h \cdot \mathcal{ODP}_n| - |\{\alpha \in \mathcal{I}_n \mid |\operatorname{Dom}(\alpha)| \leq 1\}| \\ & = & (3 \cdot 2^n - 2(n+1)) + (3 \cdot 2^n - 2(n+1)) - (n^2 + 1) \\ & = & 3 \cdot 2^{n+1} - (n+2)^2 - 1. \end{array}$$

Next, we recall that Al-Kharousi et al. proved in [3] that $\mathcal{ODP}_n$ is generated by $A = \{x_i \mid 1 \leq i \leq n\}$ (as a monoid), where

$$x_i = \begin{pmatrix} 1 & \cdots & n-i-1 & n-i+1 & \cdots & n \\ 1 & \cdots & n-i-1 & n-i+1 & \cdots & n \end{pmatrix},$$

for $1 \leq i \leq n - 2$,

$$x_{n-1} = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} \quad \text{and} \quad x_n = \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}.$$

Moreover, they also proved that A is the unique minimal (for set inclusion) generating set of $\mathcal{ODP}_n$, from which follows immediately that $\mathcal{ODP}_n$ has rank equal to n (as a monoid).

Now, since $\mathcal{ODP}_n \cup \{h\}$ generates the monoid $\mathcal{DP}_n$, it follows that $B = A \cup \{h\}$ is also a generating set of $\mathcal{DP}_n$. Moreover, it is easy to show that $x_{n-1} = h x_n h$ and $x_{n-i-1} = h x_i h$, for $1 \leq i \leq n - 2$. Therefore $C = \{h, x_n\} \cup \{x_i \mid 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor\}$ is another generating set for $\mathcal{DP}_n$. Observe that C as $\lfloor \frac{n+3}{2} \rfloor$ elements, which coincides with the rank of $\mathcal{DP}_n$ [3].

3 A presentation for $\mathcal{ODP}_n$

Consider the set $A = \{x_i \mid 1 \le i \le n\}$ as an alphabet (with n letters) and the set R formed by the following $\frac{1}{2}n^2 + \frac{1}{2}n + 3$ monoid relations:

(R_1) $x_i^2 = x_i$, $1 \le i \le n-2$;

(R_2) $x_i x_j = x_j x_i$, $1 \le i < j \le n-2$;

(R_3) $x_{n-1}x_i = x_{i+1}x_{n-1}$, $1 \le i \le n-3$;

(R_4) $x_n x_{i+1} = x_i x_n$, $1 \le i \le n-3$;

(R_5) $x_{n-1}^2 x_n = x_1 x_{n-1}$;

(R_6) $x_n x_{n-1}^2 = x_{n-1}x_{n-2}$;

(R_7) $x_n^2 x_{n-1} = x_{n-2}x_n$;

(R_8) $x_{n-1}x_n^2 = x_n x_1$;

(R_9) $x_{n-1}x_n x_{n-1} = x_{n-1}$;

(R_{10}) $x_n x_{n-1} x_n = x_n$;

(R_{11}) $x_n^n = x_1 \cdots x_{n-2}x_{n-1}x_{n-2}$;

(R_{12}) $x_n^{n+1} = x_n^n$.

This section is dedicated to prove that $\langle A \mid R\rangle$ is a presentation for the monoid $\mathcal{ODP}_n$, using the method given by Theorem 1.1.

Observe that we can easily deduce from R the following four relations, which are useful to simplify some calculations:

$$x_1 x_{n-1}^2 = x_{n-1}^2, \quad x_{n-1}^2 x_{n-2} = x_{n-1}^2, \quad x_{n-2}x_n^2 = x_n^2 \quad \text{and} \quad x_n^2 x_1 = x_n^2. \tag{1}$$

First, it is a routine matter to prove that all relations over the alphabet A from R are satisfied by the generating set A of $\mathcal{ODP}_n$ (with the natural correspondence between letters and generators).

Next, we define our set of forms for $\langle A \mid R\rangle$.

Let

$$W_{n-1} = \{x_1, \ldots, x_n, x_{n-1}x_n, x_n x_{n-1}\}$$

Notice that $|W_{n-1}| = n+2$. For $2 \le k \le n-1$, let

$W_{n-k,1} = \left\{\left(x_{\ell_1}\cdots x_{\ell_{k-1}}\right) x_i \mid 1 \le \ell_1 < \cdots < \ell_{k-1} \le n-2, \quad \ell_{k-1} < i \le n\right\}$,

$W_{n-k,2} = \left\{\left(x_{\ell_1}\cdots x_{\ell_{k-1}}\right) x_{n-1}x_n, \quad \left(x_{\ell_1}\cdots x_{\ell_{k-1}}\right) x_n x_{n-1} \mid 1 \le \ell_1 < \cdots < \ell_{k-1} \le n-2\right\}$,

$W_{n-2,3} = \{x_{n-1}x_{n-2}x_n\}$ and, for $k \ge 3$, $W_{n-k,3} = \left\{\left(x_{\ell_1}\cdots x_{\ell_{k-2}}\right) x_{n-1}x_{n-2}x_n \mid 1 \le \ell_1 < \cdots < \ell_{k-2} \le n-2\right\}$,

$W_{n-k,4} = \left\{\left(x_{\ell_1}\cdots x_{\ell_{k-i}}\right) x_{n-1}^i, \quad x_{n-1}^k \mid i \le \ell_1 < \cdots < \ell_{k-i} \le n-2, \quad 2 \le i < k\right\}$,

$W_{n-k,5} = \left\{\left(x_{\ell_1}\cdots x_{\ell_{k-i}}\right) x_n^i, \quad x_n^k \mid 1 \le \ell_1 < \cdots < \ell_{k-i} \le n-i-1, \quad 2 \le i < k\right\}$,

$W_{n-k,6} = \left\{\left(x_{\ell_1}\cdots x_{\ell_{k-i}}\right) x_{n-1}^{i-1}x_{n-i}, \quad x_{n-1}^{k-1}x_{n-k} \mid i-1 \le \ell_1 < \cdots < \ell_{k-i} \le n-2, \quad 2 \le i < k\right\}$,

$W_{n-k,7} = \left\{\left(x_{\ell_1}\cdots x_{\ell_{k-i}}\right) x_n x_1 x_n^{i-2}, \quad x_n x_1 x_n^{k-2} \mid 1 \le \ell_1 < \cdots < \ell_{k-i} \le n-i, \quad 2 \le i < k\right\}$

and

$$W_{n-k} = W_{n-k,1} \cup W_{n-k,2} \cup W_{n-k,3} \cup W_{n-k,4} \cup W_{n-k,5} \cup W_{n-k,6} \cup W_{n-k,7}.$$

Notice that, for $2 \le k \le n-1$, we have

$|W_{n-k,1}| = \binom{n-2}{k} + 2\binom{n-2}{k-1}$,

$|W_{n-k,2}| = 2\binom{n-2}{k-1}$,

$|W_{n-k,3}| = \binom{n-2}{k-2}$,

$|W_{n-k,4}| = |W_{n-k,5}| = \sum_{i=2}^{k} \binom{n-i-1}{k-i}$,

$|W_{n-k,6}| = |W_{n-k,7}| = \sum_{i=2}^{k} \binom{n-i}{k-i}$

and so $|W_{n-k}| = \binom{n-2}{k} + 2\binom{n-2}{k-1} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2} + 2\sum_{i=2}^{k} \binom{n-i-1}{k-i} + 2\sum_{i=2}^{k} \binom{n-i}{k-i} = \binom{n}{k} + 2\sum_{i=1}^{k} \binom{n-i}{k-i}$. Thus, for $1 \le k \le n-1$, we have

$$|W_k| = \binom{n}{n-k} + 2\sum_{i=1}^{n-k} \binom{n-i}{n-k-i} = \binom{n}{k} + 2\sum_{i=0}^{n-k-1} \binom{k+i}{i} = \binom{n}{k} + 2\sum_{i=k}^{n-1} \binom{i}{k}$$

and, by Gould (1.52) identity [17, page 7], i.e. $\sum_{i=k}^{m} \binom{i}{k} = \binom{m+1}{k+1}$, it follows $|W_k| = \binom{n}{k} + 2\binom{n}{k+1}$. Finally, let

$$W_0 = \{x_n^n\}$$

and

$$W = \{1\} \cup \bigcup_{k=0}^{n-1} W_k$$

Notice that

$$\begin{array}{rcl} |W| & = & 2 + \sum_{k=1}^{n-1} |W_k| \\ & = & 2 + \sum_{k=1}^{n-1} (\binom{n}{k} + 2\binom{n}{k+1}) \\ & = & 2 + \sum_{k=1}^{n-1} \binom{n}{k} + 2\sum_{k=1}^{n-1} \binom{n}{k+1} \\ & = & 2 + \sum_{k=1}^{n-1} \binom{n}{k} + 2\sum_{k=2}^{n} \binom{n}{k} \\ & = & 2 + (2^n - 1 - 1) + 2(2^n - 1 - n) \\ & = & 3 \cdot 2^n - 2(n+1) \\ & = & |\mathcal{ODP}_n|. \end{array}$$

Observe that, for $0 \le k \le n-1$, each word of W_k represents a transformation of rank k of $\mathcal{ODP}_n$.

Lemma 3.1 *W constitutes a set of forms for $\langle A \mid R\rangle$.*

Proof. As observed after Theorem 1.1 it suffices to show that for each letter $x \in A$ and for each word $w \in W$, there exists a word $w' \in W$ such that the relation $wx = w'$ is a consequence of R.

In order to perform this aim, we consider separately the word $w \in W$ in each of the subsets considered above that defines W. Namely, in W_{n-1}, $W_{n-k,r}$, for $1 \le r \le 7$ and $2 \le k \le n-1$, and W_0.

I. Let $w \in W_{n-1}$ and $j \in \{1, \ldots, n\}$. We consider five cases.

Case 1. $w \equiv x_i$, with $i \in \{1, \ldots, n-2\}$. If $i < j$ then $wx_j \equiv x_ix_j \in W_{n-2,1}$. If $j < i$, by applying a relation (R_2), we have $wx_j \equiv x_ix_j = x_jx_i \in W_{n-2,1}$. If $j = i$ then, by applying a relation (R_1), we have $wx_j \equiv x_i^2 = x_i \in W_{n-1}$.

Case 2. $w \equiv x_{n-1}$. If $j \in \{1, \dots, n-3\}$ then, by a relation (R_3), we have $wx_j \equiv x_{n-1}x_j = x_{j+1}x_{n-1} \in W_{n-2,1}$. If $j = n-2$ then $wx_j \equiv x_{n-1}x_{n-2} \in W_{n-2,6}$. If $j = n-1$ then $wx_j \equiv x_{n-1}^2 \in W_{n-2,4}$. If $j = n$ then $wx_j \equiv x_{n-1}x_n \in W_{n-1}$.

Case 3. $w \equiv x_n$. If $j = 1$ then $wx_j \equiv x_nx_1 \in W_{n-2,7}$. If $j \in \{2, \dots, n-2\}$ then, by a relation (R_4), we have $wx_j \equiv x_nx_j = x_{j-1}x_n \in W_{n-2,1}$. If $j = n-1$ then $wx_j \equiv x_nx_{n-1} \in W_{n-1}$. If $j = n$ then $wx_j \equiv x_n^2 \in W_{n-2,5}$.

Case 4. $w \equiv x_{n-1}x_n$. If $j = 1$ then, by relations (R_8) and (R_5), we have $wx_j \equiv x_{n-1}x_nx_1 = x_{n-1}^2x_n^2 = x_1x_{n-1}x_n \in W_{n-2,2}$. If $j \in \{2, \dots, n-2\}$ then, by relations (R_4) and (R_3), we have $wx_j \equiv x_{n-1}x_nx_j = x_{n-1}x_{j-1}x_n = x_jx_{n-1}x_n \in W_{n-2,2}$. If $j = n-1$ then, by the relation (R_9), we have $wx_j \equiv x_{n-1}x_nx_{n-1} = x_{n-1} \in W_{n-1}$. If $j = n$ then, by the relation (R_8), we have $wx_j \equiv x_{n-1}x_n^2 = x_nx_1 \in W_{n-2,7}$.

Case 5. $w \equiv x_nx_{n-1}$. If $j \in \{1, \dots, n-3\}$ then, by relations (R_3) and (R_4), we have $wx_j \equiv x_nx_{n-1}x_j = x_nx_{j+1}x_{n-1} = x_jx_nx_{n-1} \in W_{n-2,2}$. If $j = n-2$ then, by relations (R_6) and (R_7), we have $wx_j \equiv x_nx_{n-1}x_{n-2} = x_n^2x_{n-1}^2 = x_{n-2}x_nx_{n-1} \in W_{n-2,2}$. If $j = n-1$ then, by the relation (R_6), we have $wx_j \equiv x_nx_{n-1}^2 = x_{n-1}x_{n-2} \in W_{n-2,6}$. If $j = n$ then, by the relation (R_{10}), we have $wx_j \equiv x_nx_{n-1}x_n = x_n \in W_{n-1}$.

II. Let $w \in W_{n-k,1}$, with $2 \le k \le n-1$, and $j \in \{1, \dots, n\}$. Then, for some $1 \le \ell_1 < \cdots < \ell_{k-1} \le n-2$ and $\ell_{k-1} < i \le n$, we have $w \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_i$. We consider three cases.

Case 1. $i \le n-2$. If $i < j$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_ix_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}x_i\right) x_j \in W_{n-(k+1),1}$. If $j < i$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_ix_j = \left(x_{\ell_1} \cdots x_{\ell_t}x_jx_{\ell_{t+1}} \cdots x_{\ell_{k-1}}\right) x_i$, with $\ell_t \le j \le \ell_{t+1}$ and $1 \le t \le k-1$, by applying $k-t$ times relations (R_2). If either ℓ_t or ℓ_{t+1} is equal to j then, by a relation (R_1), we have $\left(x_{\ell_1} \cdots x_{\ell_t}x_jx_{\ell_{t+1}} \cdots x_{\ell_{k-1}}\right) x_i = w \in W_{n-k,1}$. Otherwise $\left(x_{\ell_1} \cdots x_{\ell_t}x_jx_{\ell_{t+1}} \cdots x_{\ell_{k-1}}\right) x_i \in W_{n-(k+1),1}$. If $j = i$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_i^2 = w \in W_{n-k,1}$, by applying a relation (R_1).

Case 2. $i = n-1$. If $j \in \{1, \dots, n-3\}$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_{n-1}x_j = \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_{j+1}x_{n-1}$, by applying a relation (R_3). Now, as above, by applying enough times relations from (R_2) and, for $j+1 \in \{\ell_1, \dots, \ell_{k-1}\}$, also a relation (R_1), we obtain $wx_j = w \in W_{n-k,1}$ or $wx_j = w' \in W_{n-(k+1),1}$. If $j = n-2$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_{n-1}x_{n-2} \in W_{n-(k+1),6}$. If $j = n-1$ and $\ell_1 > 1$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_{n-1}^2 \in W_{n-(k+1),4}$. If $j = n-1$ and $\ell_1 = 1$ then $wx_j \equiv \left(x_1 \cdots x_{\ell_{k-1}}\right) x_{n-1}^2 = \left(x_{\ell_2} \cdots x_{\ell_{k-1}}\right) x_1x_{n-1}^2 = \left(x_{\ell_2} \cdots x_{\ell_{k-1}}\right) x_{n-1}^2 \in W_{n-k,4}$, by first applying $k-1$ times relations from (R_2) and then the first relation of (1). If $j = n$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_{n-1}x_n \in W_{n-k,2}$.

Case 3. $i = n$. If $j = 1$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_nx_1 \in W_{n-(k+1),7}$. If $j \in \{2, \dots, n-2\}$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_nx_j = \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_{j-1}x_n$, by a relation (R_4). Again, as above, by applying enough times relations from (R_2) and, for $j-1 \in \{\ell_1, \dots, \ell_{k-1}\}$, also a relation (R_1), we obtain $wx_j = w \in W_{n-k,1}$ or $wx_j = w' \in W_{n-(k+1),1}$. If $j = n-1$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_nx_{n-1} \in W_{n-k,2}$. If $j = n$ and $\ell_{k-1} < n-2$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-1}}\right) x_n^2 \in W_{n-(k+1),5}$. If $j = n$ and $\ell_{k-1} = n-2$ then $wx_j \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-2}}x_{n-2}\right) x_n^2 \equiv \left(x_{\ell_1} \cdots x_{\ell_{k-2}}\right) x_{n-2}x_n^2 = \left(x_{\ell_1} \cdots x_{\ell_{k-2}}\right) x_n^2 \in W_{n-k,5}$, by applying the third relation of (1).

III. For $w \in W_{n-k,r}$, with $2 \le k \le n-1$ and $2 \le r \le 7$, and $j \in \{1, \dots, n\}$, similar calculations to the previous cases, mostly routine, assure us the existence of $w' \in W$ such that the relation $wx_j = w'$ is a consequence of R. In fact, we may find w' belonging to:

1. $W_{n-k,2} \cup W_{n-(k+1),2} \cup W_{n-k,1} \cup W_{n-(k+1),6} \cup W_{n-(k+1),7}$, if $r = 2$;
2. $W_{n-3,3} \cup W_{n-2,6} \cup W_{n-2,7}$, if $r = 3$ and $k = 2$;
3. $W_{n-k,3} \cup W_{n-(k+1),3} \cup W_{n-k,6} \cup W_{n-k,7}$, if $r = 3$ and $k \ge 3$;
4. $W_{n-k,4} \cup W_{n-(k+1),4} \cup W_{n-k,1} \cup W_{n-(k+1),6}$, if $r = 4$;
5. $W_{n-k,5} \cup W_{n-(k+1),5} \cup W_{n-k,1} \cup W_{n-k,4} \cup W_{n-(k+1),7}$, if $r = 5$;

6. $W_{n-k,6} \cup W_{n-(k+1),6} \cup W_{n-k,3} \cup W_0$, if $r = 6$;
7. $W_{n-k,7} \cup W_{n-(k+1),7} \cup W_{n-k,3}$, if $r = 7$.

IV. Finally, we show that $x_n^n x_j = x_n^n$ is a consequence of R, for $j \in \{1, \ldots, n\}$. We consider four cases.

Case 1. $j \in \{1, \ldots, n-3\}$. Then

$$\begin{aligned} x_n^n x_j &= x_1 \cdots x_{n-2} x_{n-1} x_{n-2} x_j \\ &= x_1 \cdots x_{n-2} x_{n-1} x_j x_{n-2} \\ &= x_1 \cdots x_{n-2} x_{j+1} x_{n-1} x_{n-2} \\ &= x_1 \cdots x_{j+1}^2 \cdots x_{n-2} x_{n-1} x_{n-2} \\ &= x_1 \cdots x_{j+1} \cdots x_{n-2} x_{n-1} x_{n-2} \\ &= x_n^n \ , \end{aligned}$$

by applying relations (R_{11}), (R_2), (R_3), (R_2), (R_1) and (R_{11}) in an orderly manner.

Case 2. $j = n-2$. By applying relations (R_{11}) and (R_1), we have

$$x_n^n x_j = x_1 \cdots x_{n-2} x_{n-1} x_{n-2}^2 = x_1 \cdots x_{n-2} x_{n-1} x_{n-2} = x_n^n.$$

Case 3. $j = n-1$. We obtain

$$\begin{aligned} x_n^n x_j &= x_n^n x_n x_{n-1} \\ &= x_1 \cdots x_{n-2} x_{n-1} x_{n-2} x_n x_{n-1} \\ &= x_1 \cdots x_{n-2} x_{n-1} x_n^2 x_{n-1}^2 \\ &= x_1 \cdots x_{n-2} x_{n-1} x_n x_{n-1} x_{n-2} \\ &= x_1 \cdots x_{n-2} x_{n-1} x_{n-2} \\ &= x_n^n \ , \end{aligned}$$

by applying relations (R_{12}), (R_{11}), (R_7), (R_6), (R_9) and (R_{11}) in an orderly manner.

Case 4. $j = n$. This is exactly relation (R_{12}).

This completes the proof of the lemma. ■

At this stage, we proved all the conditions of Theorem 1.1. Therefore, we have:

Theorem 3.2 *The monoid $\mathcal{ODP}_n$ is defined by the presentation $\langle A \mid R \rangle$, on n generators and $\frac{1}{2}n^2 + \frac{1}{2}n + 3$ relations.*

4 Presentations for $\mathcal{DP}_n$

In this section we exhibit two presentations for $\mathcal{DP}_n$. We start by construct a presentation associated to the set of generators B and then deduce a new one associated to the set of generators C. Recall that $B = A \cup \{h\} = \{x_1, \ldots, x_n, h\}$ and $C = \{h, x_n\} \cup \{x_i \mid 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor\}$.

Consider the set B as an alphabet (with $n+1$ letters) and the set $\overline{R}$ formed by all relations from R (defined in the previous section) together with the following monoid relations:

(NR_0) $h^2 = 1$;

(NR_1) $hx_{n-1} = x_n h$, $hx_n = x_{n-1} h$ and $hx_i = x_{n-i-1} h$, $1 \leq i \leq n-2$;

(NR_2) $x_n^{n-1} h = x_n x_{n-1} x_{n-2} \cdots x_2 x_1$.

Notice that $\overline{R}$ is a set of $\frac{1}{2}n^2 + \frac{3}{2}n + 5$ monoid relations over the alphabet B.

It is a routine matter to prove that all relations from $\overline{R}$ are satisfied by the generating set B of $\mathcal{DP}_n$ (with the natural correspondence between letters and generators).

Now, let $\alpha_{i,j} = \binom{i}{j} \in \mathcal{ODP}_n$, for $1 \leq i, j \leq n$. Then $x_n^{n-1} = \alpha_{n,1}$ and $\alpha_{i,j} = \alpha_{i,n}\alpha_{n,1}\alpha_{1,j} = \alpha_{i,n}x_n^{n-1}\alpha_{1,j}$, for $1 \leq i, j \leq n$. Let u_i and v_i be the words of W (defined in the previous section or even any other set of representatives of all elements of $\mathcal{ODP}_n$ over the alphabet A) that represents the elements $\alpha_{i,n}$ and $\alpha_{1,i}$ of $\mathcal{ODP}_n$, respectively, for $1 \leq i \leq n$. Hence the word $u_i x_n^{n-1} v_j$ over the alphabet A also represents the transformation $\alpha_{i,j} \in \mathcal{ODP}_n$, for $1 \leq i, j \leq n$.

Let $W_\alpha = \{u_i x_n^{n-1} v_j \mid 1 \leq i, j \leq n\} \cup W_0$ and $W_\beta = W \setminus W_\alpha$. Notice that W_α is a set of representatives for the transformations of $\mathcal{ODP}_n$ of rank less than or equal to one and x_n^{n-1} is a factor of each word of W_α. Let $\overline{W} = W \cup \{wh \mid w \in W_\beta\}$. Since also $|\overline{W}| = |W| + |W_\beta| = 2|W| - |W_\alpha| = 2|\mathcal{ODP}_n| - (n^2+1) = |\mathcal{DP}_n|$, by Theorem 1.3, we may conclude that the monoid $\mathcal{DP}_n$ is defined by the presentation $\langle B \mid \overline{R} \rangle$.

On the other hand, it is easy to show that relations (NR_1) are a consequence of (NR_0) and

$(\overline{NR}_1)$ $hx_n = x_{n-1}h$ and $hx_i = x_{n-i-1}h$, $1 \leq i \leq \lfloor\frac{n-1}{2}\rfloor$;

Moreover, within the context of relations (NR_0) and (NR_1):

1. Relations (R_1) are equivalent to relations

   $(\overline{R}_1)$ $x_i^2 = x_i$, $1 \leq i \leq \lfloor\frac{n-1}{2}\rfloor$;

2. Relations (R_3) are equivalent to relations (R_4);

3. Relations (R_5) and (R_7) are (both) equivalent to relation

   $(\overline{R}_7)$ $x_n^2 hx_n h = hx_1 hx_n$;

4. Relations (R_6) and (R_8) are (both) equivalent to relation

   $(\overline{R}_8)$ $hx_n hx_n^2 = x_n x_1$;

5. Relations (R_9) and (R_{10}) are (both) equivalent to relation

   $(\overline{R}_{10})$ $x_n hx_n hx_n = x_n$.

Therefore, if we denote by U the set of monoid relations over B formed by all relations from

$$(\overline{R}_1),\ (R_2),\ (R_4),\ (\overline{R}_7),\ (\overline{R}_8),\ (\overline{R}_{10}),\ (R_{11}),\ (R_{12}),\ (NR_0),\ (\overline{NR}_1) \text{ and } (NR_2),$$

we have:

Theorem 4.1 *The monoid $\mathcal{DP}_n$ is defined by the presentation $\langle B \mid U\rangle$, on $n+1$ generators and $\frac{1}{2}(n^2 - n + 13 - (-1)^n)$ relations.*

We finish this section, and the paper, by *removing* superfluous generators from the above presentation of $\mathcal{DP}_n$, i.e. since $x_{n-1} = hx_n h$ and $x_{n-i-1} = hx_i h$, for $1 \leq i \leq \lfloor\frac{n-1}{2}\rfloor$, we remove all letters/generators x_i, for $\frac{n-1}{2} < i \leq n-1$, replace all of their occurrences by the previous expressions in all relations of U and remove all trivial relations and all relations clearly deductible from others obtained in the process.

First, we turn our attention to relations (R_2). Since $1 \leq i < j \leq n-2$ may be decompose in

$$1 \leq i < j \leq \lfloor\frac{n-1}{2}\rfloor \quad \text{or} \quad 1 \leq i \leq \lfloor\frac{n-1}{2}\rfloor < j \leq n-2 \quad \text{or} \quad \lfloor\frac{n-1}{2}\rfloor < i < j \leq n-2\ ,$$

and, for the case $\lfloor\frac{n-1}{2}\rfloor < i < j \leq n-2$, we have $x_i x_j = x_j x_i$ if and only if $(hx_{n-i-1}h)(hx_{n-j-1}h) = (hx_{n-j-1}h)(hx_{n-i-1}h)$ if and only if $x_{n-i-1}x_{n-j-1} = x_{n-j-1}x_{n-i-1}$, and $1 \leq n-j-1 < n-i-1 \leq \lfloor\frac{n-1}{2}\rfloor$, then we obtain the following relations from (R_2):

$(\overline{R}_2)$ $x_ix_j = x_jx_i$, $1 \le i < j \le \lfloor\frac{n-1}{2}\rfloor$, and $x_ihx_{n-j-1}h = hx_{n-j-1}hx_i$, $1 \le i \le \lfloor\frac{n-1}{2}\rfloor < j \le n-2$.

Now, we consider the relations (R_4): $x_nx_{i+1} = x_ix_n$, with $1 \le i \le n-3$. For $i = \lfloor\frac{n-1}{2}\rfloor$, we get the relation $x_nhx_{\lceil\frac{n-3}{2}\rceil}h = x_{\lfloor\frac{n-1}{2}\rfloor}x_n$ and, for $\lfloor\frac{n-1}{2}\rfloor < i \le n-3$, we have $x_nhx_{n-i-2}h = hx_{n-i-1}hx_n$, i.e. $x_nhx_jh = hx_{j+1}hx_n$, with $1 \le j \le \lceil\frac{n-5}{2}\rceil$. Hence, we obtain the following relations from (R_4):

$(\overline{R}_4)$ $x_nx_{i+1} = x_ix_n$, $1 \le i \le \lfloor\frac{n-3}{2}\rfloor$, $x_nhx_{\lceil\frac{n-3}{2}\rceil}h = x_{\lfloor\frac{n-1}{2}\rfloor}x_n$ and $x_nhx_ih = hx_{i+1}hx_n$, $1 \le i \le \lceil\frac{n-5}{2}\rceil$.

Next, from (R_{11}) and $(N\!R_2)$, we obtain

$(\overline{R}_{11})$ $x_n^n = x_1\cdots x_{\lfloor\frac{n-1}{2}\rfloor}hx_{\lceil\frac{n-3}{2}\rceil}\cdots x_1x_nx_1h$,

$(\overline{N\!R}_2)$ $x_n^{n-1}h = x_nhx_nx_1\cdots x_{\lceil\frac{n-3}{2}\rceil}hx_{\lfloor\frac{n-1}{2}\rfloor}\cdots x_1$,

respectively.

Finally, it is easy to show that, from $(\overline{N\!R}_1)$, only for n odd we obtain non trivial relations and, in fact, a unique one, namely

$(\widehat{N\!R}_1)$ $hx_{\frac{n-1}{2}} = x_{\frac{n-1}{2}}h$, if n is odd.

Let us assume that $(\widehat{N\!R}_1)$ denotes the empty set for n even.

Denote by V the set of monoid relations over B formed by all relations from

$$(\overline{R}_1),\ (\overline{R}_2),\ (\overline{R}_4),\ (\overline{R}_7),\ (\overline{R}_8),\ (\overline{R}_{10}),\ (\overline{R}_{11}),\ (R_{12}),\ (N\!R_0),\ (\widehat{N\!R}_1) \text{ and } (\overline{N\!R}_2).$$

Thus, we have:

Theorem 4.2 *The monoid $\mathcal{DP}_n$ is defined by the presentation $\langle C \mid V\rangle$, on $\lfloor\frac{n+3}{2}\rfloor$ generators and $\frac{3}{8}n^2 + \frac{45}{8}$ relations, for n odd, and $\frac{3}{8}n^2 - \frac{1}{4}n + 5$ relations, for n even.*

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